WAVE MOTIONS PRODUCED IN A STRATIFIED LIQUID FROM FLOW PAST A SUBMERGED BODY

I. V. Sturova

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The integral-transform method is used to solve the linearized two- and three-dimensional problems of the wave motions produced by the flow of a density-stratified liquid past a submerged source and sink of equal intensity.

The method of [1, 2] can be used to study the problem of wave flows caused by a source and sink of equal intensity m submerged below a free surface in the uniform flow of a heavy liquid of infinite depth. The liquid is assumed to be nonviscous and incompressible and to be stratified exponentially in density. The x and z axes lie in the unperturbed free surface and are chosen so that the direction of the liquid velocity vector well upstream coincides with the x axis, and the y axis is directed vertically upwards. In the unperturbed state the density of the liquid varies with depth in a known way:

$$\rho_0 = \rho_0(0)\exp(-y/L), \quad y \leqslant 0 \tag{1}$$

We consider the two-dimensional (source and sink are linear and parallel to the z axis), and the three-dimensional cases (point source and sink).

The straight-line segment joining the source and sink is parallel to the x axis, lies at a depth h below the free surface, and has a length $2a$. The y axis passes through the center of this segment.

It is well known (see, for example, [3]) that the unbounded flow of a uniform liquid past such a source–sink combination is equivalent to flow past a closed symmetric oval in the two-dimensional case and an axially symmetric oval in the three-dimensional case. We assume that to a first approximation this result also holds for our problem, providing that the depth h is sufficiently great and that the stratification is weak.

If we are given the maximum half-width of the body R, its elongation d, and the main flow velocity U, we can find the values of $\alpha = a/R$ in the two cases by solving the equations

$$\alpha^2 + \alpha/\operatorname{arctg}\alpha = d^2, \quad m = \pi UR/\operatorname{arctg}\alpha$$
$$(d^2 - \alpha^2)^2 = d\sqrt{\alpha^2+1}, \quad m = \pi UR^2\sqrt{\alpha^2+1}/\alpha$$

The problem is a stationary one, but for the reasons given in [1] it is necessary to treat it as nonstationary and to use initial conditions. It is therefore assumed that the source and sink begin to act simultaneously at t = 0 and that their intensity remains constant for t > 0. The solution of this nonstationary problem goes over to the solution of the original stationary problem in the limit $t \to \infty$. There is as yet no rigorous mathematical proof of this fact for the present problem, and the statement is made here from physical considerations.

The equations of motion for a spatial (three-dimensional) flow are

$$\begin{gathered}\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = mH(t)\,[\delta(x+a) - \delta(x-a)]\,\delta(y+h)\,\delta(z)\\ \frac{du}{dt} = -\frac{1}{\rho}\frac{dp}{\partial x}, \quad \frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g\\ \frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z}, \quad \frac{d\rho}{dt} = 0, \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\end{gathered} \tag{2}$$

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Here u, v, w, are the components of the velocity vector in the direction of the x, y, and z axes, respectively, ρ is the density, p is the pressure, g is the acceleration due to gravity, δ is the Dirac delta function, and H(t) is the Heaviside function

$$H(t) = \begin{cases} 0, & t \leqslant 0 \\ 1, & t > 0 \end{cases}$$

At the free surface

$$F_0(x, y, z, t) = y - \eta_0(x, z, t) = 0$$
$$dF_0 / dt = 0, \quad dp / dt = 0 \tag{3}$$

It is assumed that

$$u \to U, \quad v \to 0, \quad w \to 0, \quad \rho \to \rho_0, \quad p \to p_0, \quad x^2 + y^2 + z^2 \to \infty$$
$$p_0 = -g \int_0^y \rho_0(y)\, dy \tag{4}$$

The initial conditions are

$$u = U, \; v = 0, \; w = 0, \; \rho = \rho_0, \; p = p_0, \; t \leqslant 0$$

and the time derivatives of u, v, w, ρ, and p are also equal to zero.

Putting

$$u = U + u_1, \; v = v_1, \; w = w_1, \; \rho = \rho_0(y) + \rho_1, \; p = p_0(y) + p_1$$

and assuming that the perturbations caused by the singularities are small, we can linearize (2) and (3) and reduce them to a single equation for the function v,

$$D^2 \Delta v + \frac{1}{L}\left[g\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right) - D^2 \frac{\partial v}{\partial y}\right] = m D^2 H(t) \times$$
$$\times [\delta(x + a) - \delta(x - a)]\, \delta(z) \left[\delta'(y + h) - \frac{1}{L}\delta(y + h)\right] \tag{5}$$

with the boundary condition

$$D^2 \frac{\partial v}{\partial y} - g\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right) = 0, \quad y = 0$$
$$D \equiv \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}, \quad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{6}$$

On an equal-density surface defined by the function

$$F(x, y, z, t) = y - \eta(x, z, t) = 0$$

which consists of particles of density $\rho_0(\bar{y})$, and in the absence of singularities represents the plane $y = \bar{y}$, we have the condition

$$dF / dt = 0, \; F(x, y, z, t) = 0$$

which after linearization becomes

$$D\eta = v, \; y = \bar{y} \tag{7}$$

The above equations and transformations are also valid for plane (two-dimensional) flow if we put w = 0 on the right sides of (2) and (5), remove the factor $\delta(z)$, and remember that all functions are independent of z. It is also assumed that the required functions in the boundary condition (4) are bounded as $x \to \infty$.

We introduce the dimensionless variables

$$(x_*, y_*, z_*, h_*, \eta_*, a_*) = \frac{1}{L}(x, y, z, h, \eta, a), \quad v_* = v/U, \quad t_* = Ut/L \tag{8}$$

where $Q_* = m/UL^2$ in the three-dimensional case, and $Q_* = m/UL$ in the two-dimensional case. In these variables (5) and (6) become (we drop the asterisk subscript)

$$\bar{D}^2\left(\Delta v - \frac{\partial v}{\partial y}\right) + \lambda\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right) = Q\, \bar{D}^2 H(t) \times$$
$$\times [\delta(x + a) - \delta(x - a)]\, \delta(z)\, [\delta'(y + h) - \delta(y + h)]$$
$$\bar{D}^2 \frac{\partial v}{\partial y} - \lambda\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right) = 0, \quad y = 0$$
$$\lambda = gL / U^2, \; \bar{D} \equiv \partial / \partial x + \partial / \partial t$$

Using the Fourier and Laplace transformations

$$f(\mu, y, \nu, s) = \int_0^\infty e^{-st}dt \int_{-\infty}^{\infty} e^{-i\mu x}dx \int_{-\infty}^{\infty} e^{-i\nu z}v(x, y, z, t)\,dz$$

for real μ, ν and Re$s > 0$ (for the plane case there is no Fourier transformation over z) and introducing the function

$$f = Q(\partial / \partial h - 1)\,G \tag{9}$$

we obtain the normal differential equation

$$G'' - G' - (k^2 - \lambda_1)G = \frac{2i}{s}\sin(\mu a)\,\delta(y + h) \tag{10}$$

with the boundary conditions

$$\begin{gathered} G' - \lambda_1 G = 0,\ y = 0;\ G \to 0,\ y \to -\infty \\ k^2 = \nu^2 + \mu^2,\ \lambda_1 = -\lambda k^2 / (s + i\mu)^2 \end{gathered} \tag{11}$$

The solution of (10) is

$$\begin{gathered} G = -\frac{i}{sM}\sin(\mu a)\exp[(y + h)/2] \\ \left\{\exp(-|y + h|M) - \frac{m_2 - \lambda_1}{m_1 - \lambda_1}\exp[(y - h)M]\right\}\ m_{1,2} = {}^1/_2 \pm M,\ M = (k^2 - \lambda_1 + {}^1/_4)^{1/2} \end{gathered} \tag{12}$$

The function M has four branch points (k_1, k_2, k_3, k_4) in the complex k plane. For small positive s

$$\begin{gathered} k_{1,2} = \pm b + \frac{i\lambda s}{b^2\sin^3\theta} + O(s^2), \quad k_{3,4} = \frac{is}{\sin\theta \pm 2\sqrt{\lambda}} + O(s^2) \\ b = \sqrt{\lambda/\sin^2\theta - {}^1/_4}, \quad \mu = k\sin\theta, \quad \nu = k\cos\theta \end{gathered} \tag{13}$$

We note that for conditions corresponding to the real ocean it is only values $\lambda > 1$ which are of practical interest. The cuts between the branch points are made as follows: between k_1 and k_2 in the upper half-plane and along the imaginary axis from k_3 upwards and from k_4 downwards.

Knowing G, we can find from (7) and (9) the function ξ — the Fourier and Laplace image for the function η,

$$\xi = \frac{iQ\sin(\mu a)}{sM(s + i\mu)}\exp[(\bar{y} + h)/2]\left\{\exp(-|\bar{y} + h|M)\left[\frac{1}{2} + M\,\mathrm{sign}(\bar{y} + h)\right] - m_1\frac{m_2 - \lambda_1}{m_1 - \lambda_1}\exp[(\bar{y} - h)M]\right\} \tag{14}$$

In addition to the four branch points of M, the function ξ has for $\lambda > {}^1/_2$ the two simple poles

$$n_{1,2} = \pm\lambda / \sin^2\theta + 2is / \sin\theta + O(s^2) \tag{15}$$

In the two-dimensional case it is necessary to put $\nu = 0$, and $\theta = \pi/2$ in (10)-(15).

It is difficult to proceed with the analysis for both cases at once and so we first consider the plane case (which is simpler).

Applying the inverse Fourier transformation and the limiting theorem for the Laplace transform we obtain

$$\eta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\mu x}\lim_{s\to 0+} s\xi d\mu = \frac{1}{\pi}\,\mathrm{Re}\int_0^\infty e^{i\mu x}\lim_{s\to 0+} s\xi d\mu \tag{16}$$

The last equation here follows from the symmetry of the branch points and the position of the poles of the function ξ relative to the imaginary axis and also from the fact that $\xi(-\mu, s)$ is the complex conjugate of $\xi(\mu, s)$ for real positive s.

Before writing out the final expression for $\eta(x)$ we transform to the dimensionless variables

$$\begin{gathered} (X, Y, \Omega) = \frac{1}{R}(x, \bar{y}, \eta), \quad q = m/UR, \quad \varepsilon = \frac{R}{L}, \quad H = \frac{h}{R} \\ \Lambda = gR / U^2,\ \beta = \sqrt{\varepsilon\Lambda - \varepsilon^2 / 4} \end{gathered} \tag{17}$$

Here x, $\bar{y}$, η, h are the original dimensioned variables. Carrying out the integration in (16] (the contour is similar to that in [1, 4]), using the residue theorem and the Jordan lemma, we obtain

$$\begin{aligned}\Omega &= \eta_1 + \eta_2 + \eta_3, \quad X \geqslant 0 \\ \Omega &= \eta_2, \quad X < 0\end{aligned} \tag{18}$$

Here

$$\eta_1(X, Y) = -4\gamma \sin(\Lambda X)\left(\Lambda - \frac{\varepsilon}{2}\right)\exp[\Lambda(Y - H) + H\varepsilon]$$

$$\gamma = q\frac{\sin(\Lambda\alpha)}{\Lambda}$$

is the residue at point $\mu = \Lambda$; this term describes waves with a period $2\pi/\Lambda$ caused by the presence of the free surface. The term

$$\eta_2(X, Y) = -\frac{1}{\pi}\,\mathrm{Im}\int_0^\infty e^{-k|X|}Z(\mu, B, Y)\,dk$$
$$\mu = ik, \quad B = i\sqrt{k^2 + \beta^2} \tag{19}$$

arises from the integration along the imaginary axis, is an even function of X, and is most important in the neighborhood of the singularities. Finally, the term

$$\eta_3(X, Y) = \frac{2}{\pi}\,\mathrm{Im}\int_0^\beta \sin(\mu X)Z(\mu, B, Y)\,d\mu, \qquad B = i\sqrt{\beta^2 - \mu^2} \tag{20}$$

which arises from integration along the cut on the real axis μ, describes internal waves caused by the presence of the exponential stratification ($\eta_3 = 0$ when $\varepsilon = 0$). Here

$$\begin{aligned}Z(\mu, B, Y) = {} & \frac{q}{\mu B}\sin(\mu\alpha)\exp[(Y + H)\varepsilon/2] \times \\ & \times\left\{\exp(-|Y + H|B)\left[\frac{\varepsilon}{2} + B\,\mathrm{sign}(Y + H)\right] - \right. \\ & \left. - \frac{(\varepsilon/2 + B)(\varepsilon/2 - B - \Lambda)}{\varepsilon/2 + B - \Lambda}\exp[(Y - H)B]\right\}\end{aligned}$$

From the continuity equation we can obtain the perturbation in longitudinal velocity

$$\begin{aligned}u_1/U &= \zeta_1 + \zeta_2 + \zeta_3, \quad X \geqslant 0 \\ u_1/U &= \zeta_2, \quad X < 0\end{aligned} \tag{21}$$

Here

$$\zeta_1(X, Y) = -\Lambda\eta_1(X, Y)$$

and the functions $\zeta_2(X, Y)$ and $\zeta_3(X, Y)$ are analogous to (19) and (20) with the kernel

$$\bar{Z}(\mu, B, Y) = -\frac{q}{\mu B}\sin(\mu\alpha)\left(\frac{\varepsilon}{2} + B\right)\exp[(Y + H)\varepsilon/2] \times$$
$$\times\left\{\exp[-|Y + H|B]\left(\frac{\varepsilon}{2} - B\right) - \frac{(\varepsilon/2 + B)(\varepsilon/2 - B - \Lambda)}{(\varepsilon/2 + B - \Lambda)}\exp(Y - H)B\right\}$$

We have investigated the cases d = 1, 5, and 10 and we give in Table 1 the corresponding values of α, q, and γ (the values of γ are given for $\Lambda = 0.5$). For d = 1 we have flow past a circle of radius R, and the source and sink form a dipole (the solution for a dipole in the stream of a stratified liquid was derived in [4]).

It has been pointed out in [4] that surface waves are only important at comparatively small submersions, and it can be seen from Table 1 that the amplitude of these waves increases only very slightly with the elongation of the body. In order to study the behavior of the internal waves we have carried out a numerical integration of (18) and (21) on a computer by the Simpson method for $\Lambda = 0.5$, H = 20, $\varepsilon = 5 \cdot 10^{-4}$ (with these values of Λ and H the waves caused by the presence of the free surface are negligibly small). In Figs. 1 and 2, curves 1–4 correspond to d = 10 and Y = 0, −16, −160, −576 and curves 5 and 6, to Y = −576 and d = 1, 5. It is interesting to note that although the shape of the free surface changes little in the presence of internal waves, the perturbations in the horizontal velocity on the free surface are maximal. The internal waves are attenuated very slowly with depth in the two-dimensional case. An increase in the elongation at constant R produces an almost proportional increase in internal-wave amplitude but hardly affects the phase pattern.

TABLE 1

d	α	q	γ
1	—	—	3.14
5	4.65	2.31	3.38
10	9.67	2.14	—4.25

We now turn to the three-dimensional problem. Going back to (14) and applying the inverse Fourier transformations and the limiting theorem for the Laplace transform we obtain

$$\eta(x, z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} e^{i\mu x}d\mu \int_{-\infty}^{\infty} e^{i\nu z}d\nu \lim_{s\to 0+} s\xi = \frac{1}{\pi^2}\mathrm{Re}\int_0^{\infty} e^{i\mu x}d\mu \int_0^{\infty} \cos\nu z d\nu \lim_{s\to 0+} s\xi$$

The last equation here follows from the fact that the integrand is an even function of ξ and that $\xi(-\mu, \nu, s)$ is the complex conjugate of $\xi(\mu, \nu, s)$ for real positive s.

Making the change of variables

$$\mu = k\sin\theta,\ \nu = k\cos\theta \qquad x = r\cos\varphi,\ z = r\sin\varphi \tag{22}$$

and taking the limit, we obtain

$$\eta(r,\varphi) = \frac{Q}{2\pi^2}\exp[(\bar{y}+h)/2]\ \mathrm{Re}\int_0^{\pi/2} d\theta\int_0^{\infty}\Phi(k,\theta)\,[e^{ikr\sin(\theta+\varphi)}+e^{ikr\sin(\theta-\varphi)}]\,dk$$

$$\Phi(k,\theta) = \frac{\sin(ak\sin\theta)}{B\sin\theta}\left\{\exp(-|\bar{y}+h|B)\left[\frac{1}{2}+B\,\mathrm{sign}\,(\bar{y}+h)\right] - \frac{(1/2+B)(1/2-B-\bar{\lambda})}{1/2+B-\bar{\lambda}}\exp[(\bar{y}-h)B]\right\} \tag{23}$$

$$B = (k^2-\bar{\lambda}+1/4)^{1/2}, \qquad \bar{\lambda} = \lambda/\sin^2\theta$$

The contour of integration is similar to that in the plane case and is chosen in the first or fourth quadrants of the k plane depending on the sign of sin $(\theta+\varphi)$ or sin $(\theta-\varphi)$ (Table 2).

We thus obtain the following integral representations for the equal-density surfaces:

$$\begin{aligned}\eta(r,\varphi) &= \frac{Q}{2\pi^2}\exp[(\bar{y}+h)/2]\left\{\int_0^{\pi/2} J_1\sin(\bar{\lambda}r\sin(\theta+\varphi))\,d\theta + \right.\\ &+\int_{\varphi}^{\pi/2} J_1\sin(\bar{\lambda}r\sin(\theta-\varphi))\,d\theta + \int_0^{\pi/2}d\theta\int_0^{\infty}J_2\,[e^{-kr\sin(\theta+\varphi)}+e^{-kr\sin|\theta-\varphi|}]\,dk+\\ &+\left.\int_0^{\pi/2}d\theta\int_0^b J_3\sin(kr\sin(\theta+\varphi))\,dk + \int_{\varphi}^{\pi/2}d\theta\int_0^b J_3\sin(kr\sin(\theta-\varphi))\,dk\right\} \quad 0\leqslant\varphi\leqslant\pi/2 \quad (x\geqslant 0)\\ \eta(r,\varphi) &= -\frac{Q}{2\pi^2}\exp[(\bar{y}+h)/2]\left\{\int_0^{\gamma}J_1\sin(\lambda r\sin(\theta-\gamma))\,d\theta - \right.\\ &-\int_0^{\pi/2}d\theta\int_0^{\infty}J_2\,[e^{-kr\sin|\theta-\gamma|}+e^{-kr\sin(\theta+\gamma)}]\,dk+\\ &+\left.\int_0^{\gamma}d\theta\int_0^b J_3\sin(kr\sin(\theta-\gamma))\,dk\right\}, \quad \pi/2<\varphi\leqslant\pi \quad (x<0),\ \gamma=\pi-\varphi\end{aligned} \tag{24}$$

Here

$$\begin{aligned} J_1 &= \frac{2\pi}{\sin\theta}\sin(a\bar{\lambda}\sin\theta)(1-2\bar{\lambda})\exp[(\bar{y}-h)(\bar{\lambda}-1/2)]\\ J_2 &= \frac{\mathrm{sh}\,(ak\sin\theta)}{f\sin\theta}\left\{\frac{1}{2}\sin(|\bar{y}+h|f) - f\,\mathrm{sign}\,(\bar{y}+h)\cos[(\bar{y}+h)f]+\right.\\ &+\left.\frac{\sin[(\bar{y}-h)f]}{2(k^2+\bar{\lambda}^2)}(k^2-4\bar{\lambda}k^2-3\bar{\lambda}^2+\bar{\lambda}) + f(\bar{\lambda}^2-k^2-\bar{\lambda})\frac{\cos[(\bar{y}-h)f]}{k^2+\lambda^2}\right\}\\ J_3 &= -\frac{\sin(ak\sin\theta)}{n\sin\theta}\left\{\cos[(\bar{y}+h)n] + 2n\sin[(\bar{y}+h)n] -\right.\\ &-\left.(\bar{\lambda}-3\bar{\lambda}^2-k^2+4\bar{\lambda}k^2)\frac{\cos[(\bar{y}-h)n]}{\bar{\lambda}^2-k^2} + 2n(\bar{\lambda}^2-\bar{\lambda}+k^2)\frac{\sin[(\bar{y}-h)n]}{\bar{\lambda}^2-k^2}\right\}\\ f &= \sqrt{k^2+b^2}, \quad n=\sqrt{b^2-k^2}, \quad b=\sqrt{\bar{\lambda}-1/4}\end{aligned} \tag{25}$$

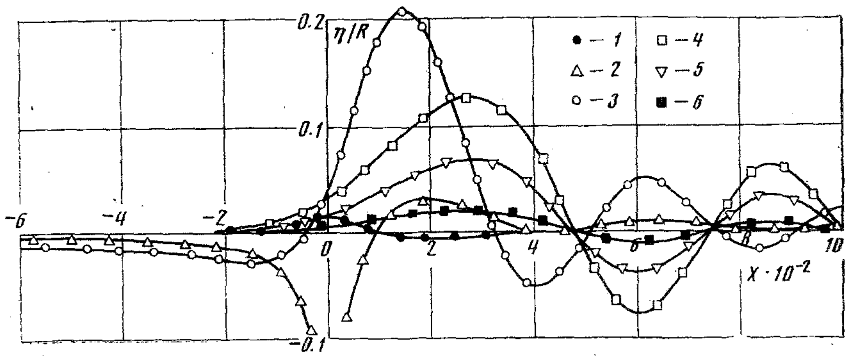

Fig. 1

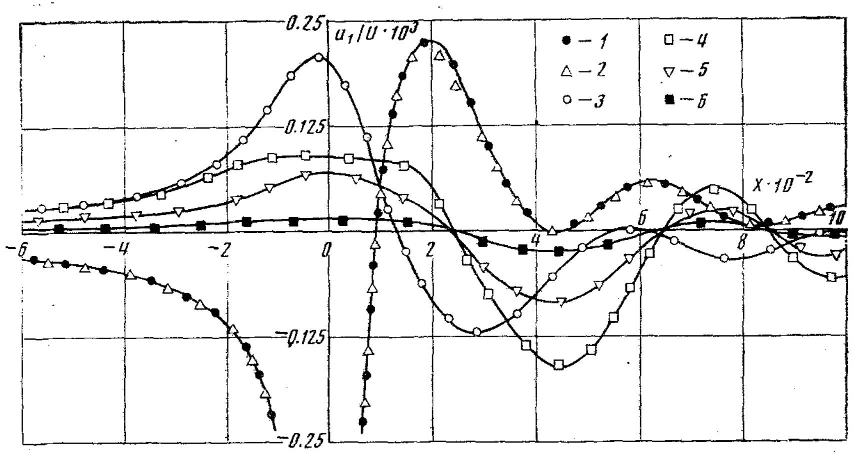

Fig. 2

TABLE 2

φ	$0<\varphi<\pi/2$		$\pi/2<\varphi<\pi$	
Quadrant	I	IV	I	IV
$e^{ikr\sin(\theta+\varphi)}$	$0<\theta<\pi/2$	—	$0<\theta<\pi-\varphi$	$\pi-\varphi<\theta<\pi/2$
$e^{ikr\sin(\theta+\varphi)}$	$\varphi<\theta<\pi/2$	$0<\theta<\varphi$	—	$0<\theta<\pi/2$

The single integrals with J_1 describe the waves caused by the presence of a free surface; these waves have been studied for a dipole in a uniform liquid in [5]. The double integrals with J_2 are even functions of x and decrease rapidly with increase in r. They describe local effects in the neighborhood of the singularities. The double integrals with J_3 represent the internal waves caused by the presence of the density stratification.

The numerical evaluation of the integrals in (24) requires large amounts of computer time and so we have only made a few calculations; in particular, for $gh/U^2 = 5$ we have determined the shape of the free surface for a flow past a dipole of a uniform liquid and a stratified liquid with $h/L = 5\cdot 10^{-3}$. The results for the two cases agree to within the numerical error (the relative accuracy of the numerical integration was 0.01), and the picture obtained is the usual wake structure. Thus, as in the two-dimensional case, the presence of weak stratification has very little effect on the shape of the free surface (see also [2, 6]).

The problem can be greatly simplified if we limit the study to a deeply submerged body and weak stratification.

In contrast to (1), we take the variation of density with depth to be

$$\rho_0(y) = \rho_0(-h)\exp[-(y+h)/L]$$

Since the preliminary calculations for the full problem (24) have shown that the terms describing the internal waves have practically no effect on the shape of the free surface, we can see from (25) that for sufficiently deep submersion the terms characterizing the surface waves will also be small, and so the free surface can be replaced by a rigid wall.

Repeating the above analysis in the dimensionless variables (8) we find that the solution of (10) with the boundary conditions (we omit the asterisk subscript)

$$G = 0\begin{cases} y = 0 \\ y \to -\infty \end{cases}$$

is

$$G = \frac{i}{sM}\sin(\mu a)e^{(y+h)/2}\left[e^{(y-h)M} - e^{-|y+h|M}\right]$$

Thus

$$\xi = \frac{iQ}{sM(s+i\mu)}\sin(\mu a)\,e^{(\bar{y}+h)/2}\left\{\left[\frac{1}{2} + M\operatorname{sign}(\bar{y}+h)\right]e^{-|\bar{y}+h|M} + m_1 e^{(\bar{y}-h)M}\right\}$$

and in (23)

$$\Phi(k,\theta) = \frac{\sin(ak\sin\theta)}{B\sin\theta}\left\{\exp(-B|\bar{y}+h|)\left[\frac{1}{2} + B\operatorname{sign}(\bar{y}+h)\right] - \left(\frac{1}{2}+B\right)\exp[(\bar{y}-h)B]\right\}$$

In the expression for $\eta(r, \varphi)$ we retain only the terms which describe the internal waves, since they are of most interest here:

$$\begin{aligned}
&\eta(r,\varphi) = \frac{Q}{\pi^2}e^{(\bar{y}+h)/2}\left\{\int_0^{\pi/2} d\theta\int_0^b J\sin(kr\sin(\theta+\varphi))\,dk + \right.\\
&\left. + \int_\varphi^{\pi/2} d\theta\int_0^b J\sin(kr\sin(\theta-\varphi))\,dk\right\}, \qquad x \geqslant 0\\
&\eta(r,\varphi) = \frac{Q}{\pi^2}e^{(\bar{y}+h)/2}\int_0^{\gamma} d\theta\int_0^b J\sin(kr\sin(\gamma-\theta))\,dk \qquad x < 0\\
&J = -\frac{\sin(ak\sin\theta)}{f\sin\theta}\left\{\frac{1}{2}\cos[(\bar{y}+h)f] + f\sin[(\bar{y}+h)f] - \right.\\
&\left. -\frac{1}{2}\cos[(\bar{y}-h)f] + f\sin[(\bar{y}-h)f]\right\}, \qquad f = \sqrt{b^2-k^2}
\end{aligned} \tag{26}$$

We make the change of variables $y' = \bar{y} + h$ and assume that the depth of submersion tends to infinity, so that in (26)

$$J = -\frac{\sin(ak\sin\theta)}{f\sin\theta}\left(\frac{1}{2}\cos fy' + f\sin fy'\right) \tag{27}$$

We now transform from the dimensionless variables (8) to new dimensionless variables [see (17)] in terms of R:

$$y_1 = (\bar{y}+h)/R,\ r_1 = r/R,\ q = m/UR^2,\ gR^2/U^2L = S,\quad \varepsilon = R/L \tag{28}$$

Here $\bar{y}$, h, $r = x/\cos\varphi$ are the original dimensioned variables, and the parameter S is inversely proportional to the Froude number. The integral expressions (26) now become [with allowance for (27)]

$$\frac{\eta}{R} = \frac{q}{\pi^2}e^{\varepsilon y_1/2}\left\{\int_0^{\pi/2} d\theta\int_0^{\bar{b}} J\sin(kr_1\sin(\theta+\varphi))\,dk + \int_\varphi^{\pi/2} d\theta\int_0^{\bar{b}} J\sin(kr_1\sin(\theta-\varphi))\,dk\right\}, \qquad x \geqslant 0$$

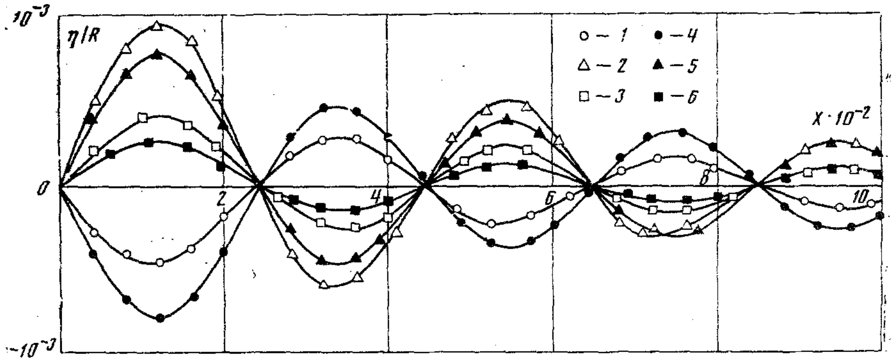

Fig. 3

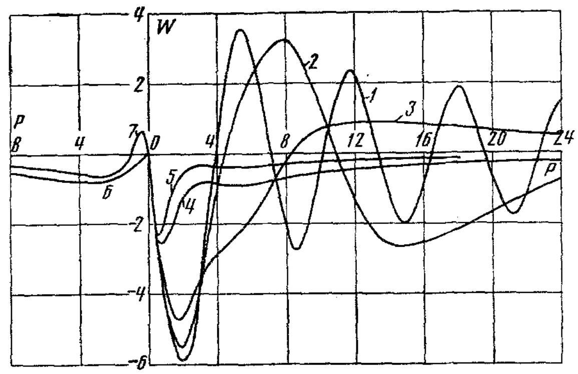

Fig. 4

$$\frac{\eta}{R} = \frac{q}{\pi^2} e^{\varepsilon y_1/2} \int_0^{\gamma} d\theta \int_0^{\bar{b}} J \sin (k r_1 \sin (\gamma - \theta))\, dk, \qquad x < 0$$

$$J = - \frac{\sin (\alpha k \sin \theta)}{n \sin \theta} \left[\frac{\varepsilon}{2} \cos (n y_1) + n \sin (n y_1) \right]$$

$$n = \sqrt{\bar{b}^2 - k^2}, \qquad \bar{b} = \sqrt{S/\sin^2 \theta - \varepsilon^2/4}$$

For $S \gg \varepsilon^2$, which corresponds to $gL/U^2 \gg 1$ — a condition which is satisfied in cases of practical interest — we can put

$$\bar{b} \approx \sqrt{S} / \sin \theta, \;\; J \approx - \sin (\alpha\, k \sin \theta) \sin (\sqrt{S / \sin^2 \theta - k^2} y_1) / \sin \theta$$

Then in the variables

$$P = r_1 \sqrt{S}, \;\; Y = y_1 \sqrt{S}$$

we obtain the universal expressions

$$W(P, \varphi, Y) = \frac{\eta}{RS \sqrt{\alpha^2 + 1}} \exp (- \varepsilon Y / 2 \sqrt{S}) =$$

$$= \frac{1}{\pi} \left\{ \int_0^{\pi/2} d\theta \int_0^{\pi/2} \bar{J} \sin \left(\frac{\sin t}{\sin \theta} P \sin (\theta + \varphi) \right) dt + \right.$$

$$\left. + \int_{\varphi}^{\pi/2} d\theta \int_0^{\pi/2} \bar{J} \sin \left(\frac{\sin t}{\sin \theta} P \sin (\theta - \varphi) \right) dt \right\}, \qquad x \geqslant 0$$

$$W(P, \varphi, Y) = \frac{1}{\pi} \int_0^{\gamma} d\theta \int_0^{\pi/2} \bar{J} \sin \left(\frac{\sin t}{\sin \theta} P \sin (\gamma - \theta) \right) dt, \qquad x < 0 \tag{29}$$

$$\bar{J} = - \frac{\sin 2t}{2 \sin^2 \theta} \sin \left(\frac{\cos t}{\sin \theta} Y \right)$$

TABLE 3

d	α	$\sqrt{\alpha^2+1}$
5	4.49	4.60
10	9.50	9.55

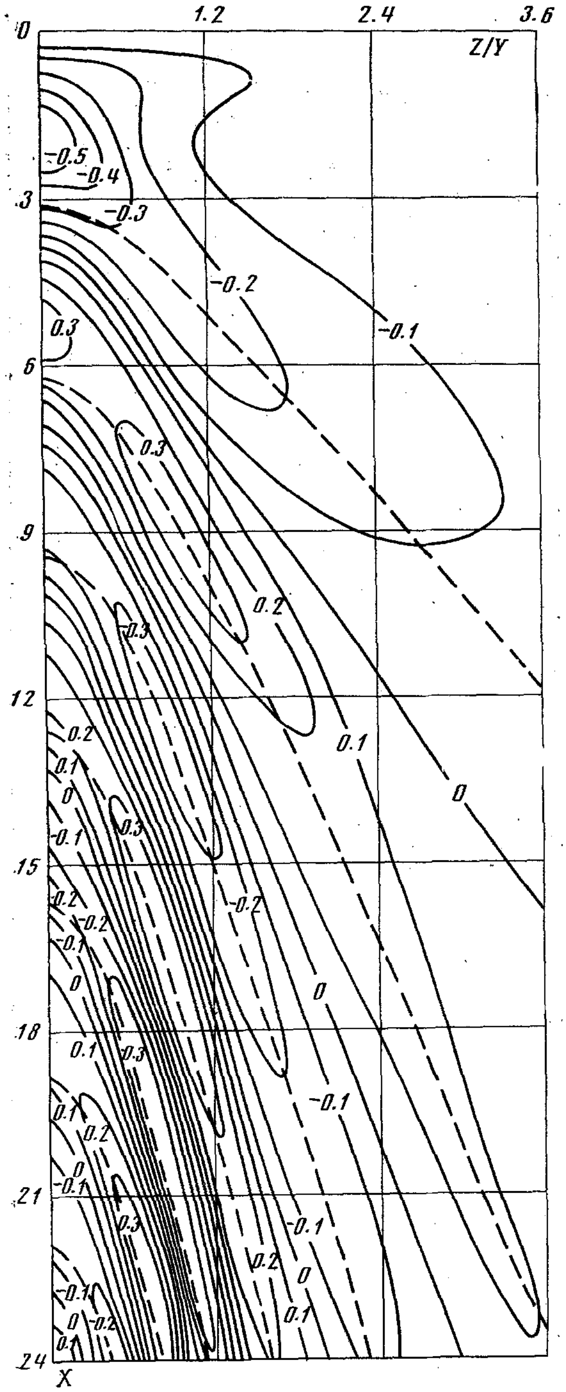

Fig. 5

in which the right sides do not depend explicitly on S. In the derivation of (29) we used the approximation

$$\sin(\alpha\sqrt{S}\sin t) \approx \alpha\sqrt{S}\sin t$$

which is valid for small values of $\alpha\sqrt{S}$.

A comparison of the solution for the full problem (24) without allowance for the terms which describe the local effects (open symbols) and for the simplified model (29) (dark symbols) is shown in Fig. 3 for a dipole with $gR/U^2 = 0.5$, $h/R = 20$, $\varepsilon = 5\cdot 10^{-4}$, $\varphi = 0$, $\bar{y}/R = -8$ (curves 1, 4), -32 (curves 2, 5), and -56 (curves 3, 6). The phase patterns are identical for the two solutions, but the amplitudes differ slightly, especially for small x. It is interesting to note that the thickness of the layer over which the internal waves are developed is much greater in the three-dimensional case.

Figure 4 shows the function $W(P, \varphi, Y)$ for $Y = 0.1$; for $x > 0$ curves 1-5 correspond to $\varphi = 0$, 0.5, 1, and 3.5° and for $x < 0$ curves 6, 7 correspond to $\gamma = 3$ and 5°. It can be seen that for $x > 0$ the function W is nonzero only in a small neighborhood of $\varphi = 0$. The effect of the body is also felt upstream, though to a much smaller extent (Fig. 4 does not show the $x < 0$ curves for which the maximum absolute value is less than 5% of the corresponding value for $x > 0$). The minimum value of Y in our calculations was 0.005. For $\varphi = 0$ the function W oscillates with P; the amplitude is proportional to $1/\sqrt{P}$, and the wavelength is equal to about 2π. In dimensioned variables this corresponds to $2\pi U\sqrt{L/g}$, which is usually very large in comparison with the wavelength of surface waves.

A rather similar conclusion was reached in [7] which dealt with internal waves caused by a moving source in a stratified liquid. However, the statement that the wake possessed axial symmetry is not confirmed by our result.

It is interesting to note that the numerical calculations indicate that to the accuracy employed the function W obeys the relationship

$$W(P, \beta\varphi, \beta Y) = \beta^{-1} W(P, \varphi, Y) \qquad (30)$$

where β is an arbitrary positive constant. By making use of (30) we can construct, even with a limited amount of data on W, a picture of the wake development in the transverse plane $x = \text{const}$ behind the body. Using (30) and the fact that the wake develops in the region of small φ values, we find that for x = const

$$W(\beta Z, \beta Y) = \beta^{-1} W(Z, Y),\ Z = P \sin \varphi$$

and it thus automatically follows that in the plane x = const the phase fronts are radial straight lines passing through the point Y = 0, Z = 0. Figure 5 shows the results for all the values of $X = P \cos \varphi$ studied and also depicts the isolines of the function $W_1 = WY$. The dashed curves were drawn from the relationship

$$Z / Y = \sqrt{(X / \pi n)^2 - 1}$$

obtained from [8] for the positions of the crests and troughs in the wave field produced from a point perturbation source in a two-dimensional problem for a linearly stratified liquid (n = 1 for the first trough, n = 2 for the first crest, and so on).

An increase in the elongation of the body, as in the plane case, produces an increase in the amplitude of the internal waves which is proportional to $\sqrt{\alpha^2 + 1}$, approximately equal to the elongation (Table 3).

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